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EQUIVALENCE OF NONLINEAR SYSTEMS TO INPUT-OUTPUT PRIME FORMS*

R. MARINO[†], W. RESPONDEK[‡], AND A. J. VAN DER SCHAFT[§]

Abstract. The problem of transforming nonlinear control systems into input-output prime forms is dealt with, using state space, static state feedback, and also output space transformations. Necessary and sufficient geometric conditions for the solvability of this problem are obtained. The results obtained generalize well-known results both on feedback linearization as well as input-output decoupling of nonlinear systems. It turns out that, from a computational point of view, the output space transformation is the crucial step, that is performed by constructing rectifying coordinates for a nested sequence of distributions on the output manifold.

Key words. equivalence, output transformation, input-output prime system, integrable distributions, input-output decoupling

AMS subject classifications. 93C10, 93B17, 58A30

1. Introduction. We consider smooth (i.e., C^∞) nonlinear systems, depending in an affine way on the inputs u_1, \dots, u_m , and having m outputs y_1, \dots, y_m

$$(1.1) \quad (\Sigma) \quad \begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^m g_j(x)u_j, & u &= (u_1, \dots, u_m) \in \mathbb{R}^m, \\ y_j &= h_j(x), & j &= 1, \dots, m \end{aligned}$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are local coordinates for the state space manifold M and for the output space manifold Y , respectively. We assume throughout the existence of an equilibrium point $x_0 \in M$ such that $f(x_0) = 0$ and $h(x_0) = 0$. (All results can be adapted to the case $f(x_0) \neq 0$ and/or $h(x_0) \neq 0$; see Remark 2 after the proof of Theorem 6.) Our analysis will be mainly of a local nature (see, however, Theorem 10 and Corollary 11 for global extensions), i.e., we firstly study the system in neighborhoods $V_{x_0} \subset M$ and $W_{y_0} \subset Y$, where $y_0 = h(x_0)$. We also assume throughout that M and Y are connected, and that $\text{rank } dh(x)$, with $h = (h_1, \dots, h_m)$, equals m in V_{x_0} , and that the dimension of the distribution $G(x) := \text{span } \{g_1(x), \dots, g_m(x)\}$ is m in V_{x_0} . Note that we are restricting ourselves entirely to square systems, i.e., the number of inputs equals the number of outputs.

We address the (local) equivalence of Σ to prime (linear) systems, and to input-output prime (linear) systems. We use the following notion of equivalence.

DEFINITION 1. Consider two systems Σ_1, Σ_2 defined on $(M_1, Y_1), (M_2, Y_2)$ with equilibrium points $x_{01} \in M_1, x_{02} \in M_2$, respectively. We say that Σ_1 is *locally equivalent* to Σ_2 , around x_{01} and x_{02} , if there exist:

- (i) Neighborhoods $V_{x_{01}} \subset M_1, V_{x_{02}} \subset M_2$ and a diffeomorphism $\varphi : V_{x_{01}} \rightarrow V_{x_{02}}$ satisfying $\varphi(x_{01}) = x_{02}$;

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- (ii) a nonsingular state feedback $u = \alpha(x) + \beta(x)v$ defined on $V_{x_{01}}$ such that $\alpha(x_{01}) = 0$ and $\det \beta(x) \neq 0$;
 - (iii) neighborhoods $W_{y_{01}} \subset Y_1, W_{y_{02}} \subset Y_2$ of $y_{01} = h^1(x_{01})$ and $y_{02} = h^2(x_{02})$, where h^1 and h^2 denote the output maps of Σ_1 and Σ_2 respectively, and a diffeomorphism $\psi : W_{y_{01}} \rightarrow W_{y_{02}}$ satisfying $\psi(y_{01}) = y_{02}$,
- such that the transformation of Σ_1 under $(\varphi, (\alpha, \beta), \psi)$ equals Σ_2 on the specified neighborhoods.

We recall from [Mo] (see also [He]) the notion of linear prime system.

DEFINITION 2. A system Σ is called a (linear) *prime system* if it is of the form

$$(1.2) \quad \begin{aligned} & y_i = x_{i1} \\ (P) \quad & \dot{x}_{i1} = x_{i2} \quad i = 1, \dots, m \\ & \vdots \\ & \dot{x}_{i\kappa_i} = u_i \end{aligned}$$

where $x = (x_{11}, \dots, x_{1\kappa_1}, \dots, x_{m1}, \dots, x_{m\kappa_m}) \in M = \mathbb{R}^n, n = \sum_{i=1}^m \kappa_i$, for some integers $\kappa_1, \dots, \kappa_m$, and $y = (y_1, \dots, y_m) \in Y = \mathbb{R}^m$. The integers $\kappa_1, \dots, \kappa_m$ equal the *orders of the zeros at infinity* of the system or the *relative degrees*, as well as the *controllability* or *observability indices*.

More generally we define input-output prime systems.

DEFINITION 3. A system Σ is called an *input-output prime system* if it is of the form

$$(1.3a) \quad \begin{aligned} & y_i = x_{i1} \\ (I - O - P) \quad & \dot{x}_{i1} = x_{i2} \quad i = 1, \dots, m \\ & \vdots \\ & \dot{x}_{i\mu_i} = u_i \end{aligned}$$

$$(1.3b) \quad \dot{z} = a(z, x) + \sum_{j=1}^m b_j(z, x)u_j, \quad a(z_0, 0) = 0,$$

where $y = (y_1, \dots, y_m) \in Y = \mathbb{R}^m$, and where the state space manifold M has the following special structure. There exists a surjective submersion $\pi : M \rightarrow \mathbb{R}^\mu, \mu := \sum_{i=1}^m \mu_i$, with $x = (x_{11}, \dots, x_{m\mu_m}) \in \mathbb{R}^\mu$, and z being complementary local coordinates for M . The integers μ_1, \dots, μ_m equal the *orders of the zeros at infinity* or the *relative degrees* of the system, as well as the *observability indices*.

Remark. Observe that the relative degrees are *not* invariant in our problem because we allow for output transformations (see the Example preceding Algorithm 7); *nor* are the observability indices since they can be changed by feedback. However, the structure at infinity *does* remain unchanged under the considered transformations, and thus this is the right concept to describe the μ_i 's as invariants in our problem. Here, the structure of infinity can be defined either geometrically using the V^* -algorithm [NS], [Is2], or by means of dynamic extension [M] since, for input-output prime systems (and their equivalents), both definitions coincide.

We will also be interested in *input-output prime systems of special form*

$$(1.4a) \quad \begin{aligned} y_i &= x_{i1} \\ (I - O - P - S) \dot{x}_{i1} &= x_{i2} \quad i = 1, \dots, m \end{aligned}$$

$$(1.4b) \quad \begin{aligned} &\vdots \\ \dot{x}_{i\mu_i} &= u_i \\ \dot{z} &= a(z, y) \end{aligned}$$

with the same specifications as in Definition 3, the difference being that the z -dynamics are only driven by the outputs $y = (y_1, \dots, y_m)$.

The main results of the paper are concerned with identifying, via necessary and sufficient geometric conditions, those nonlinear systems Σ which are locally equivalent to prime systems (Theorem 4), to input-output prime systems (Theorem 6), and to input-output prime systems of special form (Proposition 8). Theorem 10 and Corollary 11 deal with global equivalence issues. The results obtained generalize well-known results both on normal forms for input-output decouplable systems as well as on feedback linearization of systems with no outputs, as we will now briefly indicate.

If outputs are not considered in Σ , and therefore output change of coordinates (iii) is omitted in Definition 1, the problem of local equivalence with prime systems becomes the well-known local *feedback linearization* problem, i.e., local feedback equivalence into linear (Brunovsky) canonical forms

$$(1.5) \quad (B) \quad \begin{aligned} \dot{x}_{i1} &= x_{i2} \\ &\vdots \\ \dot{x}_{i\kappa_i} &= u_i \end{aligned} \quad i = 1, \dots, m,$$

which was completely solved in [JR] and [HSM]. The solution to this problem is a generalization of a linear result of Brunovsky [Br], stating that any controllable linear system

$$(1.6) \quad \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad \text{rank } B = m,$$

can be transformed into (B) by the action of the linear feedback group taking the pair (A, B) into $(T(A + BF)T^{-1}, TBG)$ for a linear state space change of coordinates $\tilde{x} = Tx$ and a linear feedback $u = Fx + Gv, \det G \neq 0$. The set of indices $(\kappa_1, \dots, \kappa_m)$, called *controllability indices*, is uniquely associated with (6) and forms a complete set of invariants under the action of the linear feedback group (see also [Wo]). In [Mo] Morse enlarges this group by allowing also for linear output space change of coordinates $\tilde{y} = Hy$, and gives necessary and sufficient conditions for a linear system

$$(1.7) \quad (L) \quad \begin{aligned} \dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^m, \\ y &= Cx, \quad \text{rank } B = \text{rank } C = m, \end{aligned}$$

to be transformed into a prime system (P) given by (1.2) by the action of the group taking (A, B, C) into $(T(A + BF)T^{-1}, TBG, HCT^{-1})$. We generalize this result of Morse to nonlinear systems Σ in Theorem 4, on the basis of the local feedback linearization theorem [JR], [HSM]. We remark that nonlinear output change of coordinates was introduced in [KR] in the study of asymptotic observers. Furthermore, the problem of local feedback

equivalence (with no output change of coordinates) of Σ to a linear system was studied and solved in [CIRT].

The problem of (local) feedback equivalence, *without* output change of coordinates, of a nonlinear system Σ to an input-output prime system $I - O - P$ has been solved in [IKGM]. Indeed, this problem amounts to the (local) nonlinear *input-output decoupling problem*, as dealt with in [SR], [Fr], and [Si]. The basic tool is the *decoupling matrix*, which generalizes to nonlinear systems the Falb–Wolovich matrix [FW], used in input-output decoupling of linear systems (L). In fact a necessary and sufficient condition for Σ to be input-output decouplable around x_0 is that its decoupling matrix is nonsingular in a neighborhood of x_0 . We note that the problem of local equivalence of Σ to $(I - O - P)$ studied in the present paper can be rephrased in this latter terminology as finding a (local) output transformation $\tilde{y} = \psi(y)$ such that Σ , with the resulting *transformed* output functions $\tilde{h}_1 = \psi_1 \circ h, \dots, \tilde{h}_m = \psi_m \circ h$, is locally input-output decouplable. Finally, (local and global) feedback equivalence with no output change of coordinates of Σ into input-output prime systems of special form $(I - O - P - S)$ has been dealt with in [BI], while for linear systems (L) equivalence to $(I - O - P)$ *implies* equivalence to $(I - O - P - S)$, as was implicitly derived in [Mo] (see Remark 2 after Proposition 8).

The results obtained are useful for control applications in the following sense. It is well known (see, e.g., [Is], [NvdS]) that many nonlinear control problems are relatively easily attacked for input-output decouplable systems. Now, in many of these control problems output transformations are naturally allowed, and thus our results enable us to treat in a similar way a class of nonlinear control systems which properly *contains* the input-output decouplable systems. One obvious example of a control problem which naturally *does* allow for output transformations is the (asymptotic) output tracking problem by static state feedback control (see the example after Theorem 6).

2. Main results. Let us first recall the definitions of the following sequences of distributions for a nonlinear system Σ :

$$(2.1) \quad \begin{aligned} G_1 &:= G := \text{span}\{g_1, \dots, g_m\} \\ G_{i+1} &:= G_i + [f, G_i], \quad i = 1, 2, \dots \end{aligned}$$

$$(2.2) \quad \begin{aligned} S_1 &:= G, \\ S_{i+1} &:= S_i + [f, S_i \cap \ker dh] + \sum_{j=1}^m [g_j, S_i \cap \ker dh], \quad i = 1, 2, \dots \end{aligned}$$

$$S^* := \bigcup_{i \geq 1} S_i.$$

The distributions G_i were introduced in [JR] in the study of the feedback linearization problem, while the algorithm (2.2) and the definition of S^* is taken from [IKGM] (with the difference that, in [IKGM], S_i in the right-hand side of (2.2) is replaced by its involutive closure; see, however, conditions (i), (iii) of Theorem 4). S^* , the smallest conditioned invariant distribution containing G , enjoys the property (see [IKGM])

$$(2.3) \quad \begin{aligned} [f, S^* \cap \ker dh] &\subset S^*, \\ [g_j, S^* \cap \ker dh] &\subset S^*, \quad j = 1, \dots, m, \end{aligned}$$

and is a generalization of the notion of the smallest conditioned invariant subspace containing $\text{Im } B$, as introduced in [BM] for a linear system (L). If the distributions $S_i, i = 0, 1, \dots$, all have constant dimension, then there exists an integer $i^* \leq n$ such that $S_{i^*} = S^*$.

Following [IKGM] we also recall the construction (if it exists) of V^* , the largest locally controlled invariant distribution contained in $\ker dh$ (see also [Hi], and for the linear case [Wo], [BM]). Define the sequence of codistributions

$$(2.4) \quad \begin{aligned} P_1 &:= dh \\ P_{i+1} &:= P_i + L_f(P_i \cap \text{ann}G) + \sum_{j=1}^m L_{g_j}(P_i \cap \text{ann}G), \quad i = 1, 2, \dots \\ P^* &:= \bigcup_{i \geq 1} P_i \end{aligned}$$

where L_f, L_{g_j} denote Lie derivatives, and $\text{ann}G$ is the codistribution annihilating G . Then the distribution V^* is the kernel of the codistribution P^* , i.e., $V^* = \ker P^*$ (see, e.g., [Is], [NvdS]).

We finally recall the definition of *characteristic indices* (or *relative degrees*) ρ_i , and of the *decoupling matrix*. For $i = 1, \dots, m$, ρ_i is defined by

$$(2.5) \quad \begin{aligned} L_{g_j} L_f^k h_i(x) &= 0, \quad k = 0, 1, \dots, \rho_i - 2, j = 1, \dots, m, \quad \text{for all } x \in V_{x_0} \\ L_{g_j} L_f^{\rho_i - 1} h_i(x) &\neq 0, \quad \text{for some } j \in \{1, \dots, m\} \quad \text{and } x \in V_{x_0}. \end{aligned}$$

If $\rho_i < \infty, i = 1, \dots, m$, the decoupling matrix $D(x)$ is defined as

$$(2.6) \quad D(x) = \left(L_{g_j} L_f^{\rho_i - 1} h_i(x) \right)_{i,j=1,\dots,m}.$$

We now come to our first main theorem.

THEOREM 4. *Consider a nonlinear system Σ with equilibrium x_0 . Σ is locally equivalent to a prime system (P) with equilibrium 0, if and only if the following conditions are satisfied in a neighborhood of x_0 :*

- (i) G_i is involutive and of constant dimension, $i = 1, \dots, n-1$;
- (ii) $G_n = TM$;
- (iii) $G_i = S_i, i = 1, 2, \dots, n$;
- (iv) $G_i + \ker dh$ is involutive and of constant dimension, $i = 1, \dots, n-1$.

Remark 1. Theorem 4 generalizes and clarifies the following result of Morse ([Mo, Thm. 3.1]): The system (L) , i.e., the triple (A, B, C) , is transformable by $(T, (F, G), H) : (A, B, C) \rightarrow (T(A + BF)T^{-1}, TBG, HCT^{-1})$ into a prime system (P) if and only if:

- (i)' $V^* = 0$;
- (ii)' $G_n = \text{Im}(B, AB, \dots, A^{n-1}B) = \mathbb{R}^n$;
- (iii)' $G_i = S_i, i = 1, \dots, n$.

Conditions (i) and (iv) of Theorem 4 are always satisfied for linear systems, while they are crucial integrability conditions in the nonlinear case. Conditions (ii)' and (iii)' of Morse are specializations of conditions (ii) and (iii) of Theorem 4 to the linear case. Condition (i), i.e., $V^* = 0$, is redundant; it is implied by conditions (ii) and (iii). In fact the proof that we will give is entirely different from Morse's and enables us to point out the redundancy of the condition $V^* = 0$ in the original statement of Morse.

Remark 2. Conditions (i) and (ii) are the necessary and sufficient conditions given in [JR] for the system Σ without outputs to be locally feedback equivalent to a linear system in Brunovsky form (B).

Remark 3. While Remarks 1 and 2 clarify the necessity of conditions (i), (ii), and (iii), we may wonder if condition (iv) is not redundant, since already condition (iii) enforces a rather strong compatibility between G_i and $\ker dh$. However, the following example shows

that for $\rho > 2$ condition (iv) is *not* implied by conditions (i), (ii), and (iii). Consider the system

$$(2.7) \quad \begin{aligned} \dot{x}_1 &= u_1, & y_1 &= x_1 \\ \dot{x}_2 &= x_3, & y_2 &= x_2 + x_1 x_5 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= x_5, & y_3 &= x_4 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_3. \end{aligned}$$

We easily compute

$$(2.8) \quad \begin{aligned} G_1 &= S_1 = \text{span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_6} \right\} \\ \ker dh &= \text{span} \left\{ \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} - x_1 \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6} \right\} \\ G_2 &= S_2 = \text{span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\} \\ G_3 &= S_3 = T\mathbb{R}^6 \\ G_1 + \ker dh &= \text{span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} - x_1 \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6} \right\} \end{aligned}$$

and thus conditions (i), (ii), and (iii) are satisfied, while condition (iv) fails since $G_1 + \ker dh$ is not involutive; in fact,

$$(2.9) \quad \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5} - x_1 \frac{\partial}{\partial x_2} \right] = -\frac{\partial}{\partial x_2} \notin G_1 + \ker dh.$$

It follows that (2.7) is *not* locally equivalent to a prime system (P).

Remark 4. It is easy to see that if a nonlinear system (Σ) with equilibrium x_0 is locally equivalent to a prime system, then its linearization at x_0 , namely, $(\bar{x} = x - x_0, \bar{y} = y - h(x_0))$

$$\begin{aligned} \dot{\bar{x}} &= \frac{\partial f}{\partial x}(x_0)\bar{x} + \sum_{j=1}^m g_j(x_0)u_j \\ \bar{y} &= \frac{\partial h}{\partial x}(x_0)\bar{x} \end{aligned}$$

is also equivalent to a prime system. The converse may be false as the following example shows:

$$\begin{aligned} \dot{x}_1 &= u_1, & y_1 &= x_1 \\ \dot{x}_2 &= x_3 + (1 - e^{x_3})u_1, & y_2 &= x_2 \\ \dot{x}_3 &= u_2. \end{aligned}$$

In this case the distribution $G_1 = \text{span}\{(\partial/\partial x_1) + (1 - e^{x_3})(\partial/\partial x_2), (\partial/\partial x_3)\}$ is not involutive, so that condition (i) of Theorem 4 is violated. On the other hand the system linearized at the origin is obviously a prime system.

Before giving the proof of Theorem 4, we first recall a lemma which clarifies the meaning of condition (iv). Let $h : M \rightarrow Y$ be such that $\text{rank } dh(x) = m = \dim Y$ on a neighborhood V_{x_0} . Then $W_{y_0} := h(V_{x_0})$ is a neighborhood of $y_0 = h(x_0)$ in Y .

Furthermore let D be a distribution on M . Then D is said to be *projectable* by h on V_{x_0} if, for all x_1, x_2 in V_{x_0} , we have

$$(2.10) \quad h(x_1) = h(x_2) \Rightarrow \frac{\partial h}{\partial x}(x_1)(D(x_1)) = \frac{\partial h}{\partial x}(x_2)(D(x_2)).$$

If D is projectable by h on V_{x_0} , then we define h_*D as the following distribution on W_{y_0} :

$$(2.11) \quad (h_*D)(y) = \partial h / \partial x(x)(D(x)), \quad \text{with } x \in h^{-1}(y) \cap V_{x_0}, \quad y \in W_{y_0}.$$

(For the problem of projecting distributions see also [J].)

LEMMA 5. *Let h be such that $\text{rank } dh(x) = \dim Y$ on V_{x_0} , and let D be involutive and constant dimensional on V_{x_0} . Then D is projectable by h in V_{x_0} to a constant dimensional and involutive distribution h_*D on W_{y_0} if and only if $D + \ker dh$ is involutive and of constant dimension on V_{x_0} .*

Proof. First notice that by the Rank Theorem [Sp] we can take local coordinates $x = (x^1, x^2)$ on V_{x_0} such that $h(x^1, x^2) = x^1$. Then it follows from [vdS] (see also [NvdS, Lem. 14.3]) that if $D + \ker dh$ is involutive and of constant dimension then D is projectable and h_*D is involutive and of constant dimension. Conversely (see [vdS]) it follows trivially that if D is projectable then $D + \ker dh$ is involutive (indeed $D + \ker dh$ is of the form $\text{span}\{k(x^1)(\partial/\partial x^1) + \text{span}\{\partial/\partial x^2\}$ for some k). Furthermore, if h_*D is constant dimensional then $D + \ker dh$ is constant dimensional. \square

Proof of Theorem 4 (only if). First we note from (2.2) that the definition of S_i is invariant under feedback and output transformations. Suppose that Σ is locally equivalent to (P) . Clearly, (P) satisfies conditions (i)–(iv). It follows that also the definition of G_1 , and inductively of $G_i, i > 1$, is invariant under feedback, and thus conditions (i)–(iv) are feedback invariant. Thus we can conclude that conditions (i)–(iv) are also satisfied for Σ .

(If.) By virtue of (i) and (iii) it follows that S_{i+1} is alternatively given as

$$(2.12) \quad S_{i+1} = S_i + [f, S_i \cap \ker dh], \quad i = 1, 2, \dots$$

since $[g_j, S_i \cap \ker dh] = [g_j, G_i \cap \ker dh] \subset G_i = S_i, j = 1, \dots, m$.

By conditions (i), (iii), (iv) and Lemma 5,

$$(2.13) \quad H_i := h_*G_i = h_*S_i, \quad i = 1, \dots, n$$

are well-defined involutive and constant-dimensional distributions on a neighborhood $W_{y_0} \subset Y$, while by (ii) $H_n = TY$. Obviously $H_1 \subset H_2 \subset \dots \subset H_n$. It follows that we can define integers

$$(2.14) \quad \kappa_1 > \kappa_2 > \dots > \kappa_r > 0$$

such that

$$(2.15) \quad 0 = H_1 = \dots = H_{\kappa_r-1} \subsetneq H_{\kappa_r} = \dots = H_{\kappa_1-1} \subsetneq H_{\kappa_1} = TY.$$

For ease of notation we will assume that

$$(2.16) \quad \dim H_{\kappa_i} = \dim H_{\kappa_i-1} + 1, \quad i = 1, \dots, m$$

implying that $r = m$, and

$$(2.17) \quad \kappa_1 > \kappa_2 > \dots > \kappa_m > 0, \quad \dim H_{\kappa_i} = m - i + 1, i = 1, \dots, m.$$

(Later on we will conclude that $\kappa_1, \dots, \kappa_m$ are actually the controllability indices of Σ .)

Invoking the generalized Frobenius theorem for the nested sequence of distributions $H_{\kappa_m} \subset H_{\kappa_{m-1}} \subset \dots \subset H_{\kappa_1}$ (see [JR], [NvdS]), we can choose locally about y_0 in $W_{y_0} m$ independent functions

$$(2.18) \quad \psi_1, \dots, \psi_m$$

such that

$$(2.19a) \quad \begin{aligned} \langle d\psi_1, H_{\kappa_1-1} \rangle &= 0 \\ \langle d\psi_2, H_{\kappa_2-1} \rangle &= 0 \\ &\vdots \\ \langle d\psi_{m-1}, H_{\kappa_{m-1}-1} \rangle &= 0 \end{aligned}$$

while

$$(2.19b) \quad \begin{aligned} \langle d\psi_1, H_{\kappa_1} \rangle(x_0) &\neq 0 \\ \langle d\psi_2, H_{\kappa_2} \rangle(x_0) &\neq 0 \\ &\vdots \\ \langle d\psi_m, H_{\kappa_m} \rangle(x_0) &\neq 0. \end{aligned}$$

In the new local coordinates of the output manifold Y , given by

$$(2.20) \quad \tilde{y}_i := \psi_i(y), \quad i = 1, \dots, m,$$

we obviously have

$$(2.21) \quad H_{\kappa_i} = \text{span} \left\{ \frac{\partial}{\partial \tilde{y}_m}, \dots, \frac{\partial}{\partial \tilde{y}_i} \right\}, \quad i = 1, \dots, m.$$

If we define new output functions

$$(2.22) \quad \tilde{h}_i := \psi_i \circ h, \quad i = 1, \dots, m$$

and recall that $H_i = h_* G_i$ (cf. (2.13)), then (2.19a) yields

$$\begin{aligned}
 \langle d\tilde{h}_1, G_{\kappa_1-1} \rangle &= 0 \\
 \langle d\tilde{h}_2, G_{\kappa_2-1} \rangle &= 0 \\
 (2.23a) \quad &\vdots \\
 \langle d\tilde{h}_{m-1}, G_{\kappa_{m-1}-1} \rangle &= 0
 \end{aligned}$$

while by (2.19b),

$$\begin{aligned}
 \langle d\tilde{h}_1, G_{\kappa_1} \rangle(x_0) &\neq 0 \\
 \langle d\tilde{h}_2, G_{\kappa_2} \rangle(x_0) &\neq 0 \\
 (2.23b) \quad &\vdots \\
 \langle d\tilde{h}_m, G_{\kappa_m} \rangle(x_0) &\neq 0.
 \end{aligned}$$

Let us now compute the decoupling matrix $\tilde{D}(x)$ (cf. (2.6)) of Σ with the newly defined output functions $\tilde{h}_1, \dots, \tilde{h}_m$ (cf. (2.22)). It readily follows from (2.23), and the Leibniz rule, i.e.,

$$(2.24) \quad L_X \langle d\varphi, Y \rangle = \langle dL_X \varphi, Y \rangle + \langle d\varphi, ad_X Y \rangle$$

for any two vector fields X, Y and function φ , that $\tilde{D}(x)$ is given as (see, e.g., [Is], [NvdS])

$$(2.25) \quad \tilde{D}(x) = \begin{pmatrix} (-1)^{(\kappa_1-1)} \langle d\tilde{h}_1, ad_f^{\kappa_1-1} g_1 \rangle(x) \dots (-1)^{(\kappa_1-1)} \langle d\tilde{h}_1, ad_f^{\kappa_1-1} g_m \rangle(x) \\ \vdots \quad \quad \quad \vdots \\ (-1)^{(\kappa_m-1)} \langle d\tilde{h}_m, ad_f^{\kappa_m-1} g_1 \rangle(x) \dots (-1)^{(\kappa_m-1)} \langle d\tilde{h}_m, ad_f^{\kappa_m-1} g_m \rangle(x) \end{pmatrix}.$$

We now make the following claim.

CLAIM. $\tilde{D}(x)$ is nonsingular in a neighborhood U_{x_0} of $x_0 \in M$.

Once this claim has been proved the rest of the proof of Theorem 4 follows easily. Indeed by the theory of input-output decoupling (see, e.g., [IKGM], [Is], [NvdS]) the functions

$$(2.26a) \quad (\tilde{h}_1, \dots, L_f^{\kappa_1-1} \tilde{h}_1, \dots, \tilde{h}_m, \dots, L_f^{\kappa_m-1} \tilde{h}_m) \\ =: (x_{11}, \dots, x_{1\kappa_1}, \dots, x_{m1}, \dots, x_{m\kappa_m})$$

are independent on U_{x_0} , and the state feedback

$$(2.26b) \quad u = -\tilde{D}^{-1}(x) \begin{pmatrix} L_f^{\kappa_1} \tilde{h}_1(x) \\ \vdots \\ L_f^{\kappa_m} \tilde{h}_m(x) \end{pmatrix} + \tilde{D}^{-1}(x)v$$

brings the system into the form

$$(2.27) \quad \begin{aligned} \dot{\tilde{y}}_i &= x_{i1} \\ \dot{x}_{i1} &= x_{i2} \quad i = 1, \dots, m, \\ &\vdots \\ \dot{x}_{i\kappa_i} &= v_i \\ \dot{z} &= a(z, x) + b(z, x)v \end{aligned}$$

(where $z \in \mathbb{R}^{n-(\kappa_1+\dots+\kappa_m)}$ are additional coordinates).

Furthermore, it is immediately seen that the S^* -algorithm (cf. (2.2)) applied to (2.27) yields $\dim S^* = \kappa_1 + \dots + \kappa_m$. Then because of feedback invariance of S_i and (ii), i.e., $S_n = G_n = TM$, it follows that $\kappa_1 + \dots + \kappa_m = n$, and thus the z -part in (2.27) is void, implying that Σ with the newly defined output functions $\tilde{h}_1, \dots, \tilde{h}_m$ is feedback equivalent to a prime system (P) , with controllability indices $\kappa_1, \dots, \kappa_m$.

Proof of the claim. We use the following induction argument.

Step 1. Consider \tilde{h}_1 . By (2.23) there exists some $i \in \{1, \dots, m\}$ such that $\langle d\tilde{h}_1, ad_f^{\kappa_1-1} g_i \rangle(x_0) \neq 0$. By relabeling g_1, \dots, g_m , if necessary, we may thus assume that

$$(2.28) \quad \langle d\tilde{h}_1, ad_f^{\kappa_1-1} g_1 \rangle(x_0) \neq 0.$$

Define the functions $\beta_i^1(x) := \langle d\tilde{h}_1, ad_f^{\kappa_1-1} g_i \rangle(x)$, $i = 1, \dots, m$, and put locally about x_0

$$(2.29) \quad \tilde{g}_i := g_i - \frac{\beta_i^1}{\beta_1^1} g_1, \quad i = 2, \dots, m$$

(observe that by (2.28) $\beta_1^1 \neq 0$ locally about x_0). Then, because

$$(2.30) \quad ad_f^{\kappa_1-1} \tilde{g}_i = ad_f^{\kappa_1-1} g_i - \frac{\beta_i^1}{\beta_1^1} ad_f^{\kappa_1-1} g_1 \pmod{G_{\kappa_1-1}},$$

the transformed input vectorfields $\tilde{g}_2, \dots, \tilde{g}_m$ satisfy

$$(2.31) \quad \langle d\tilde{h}_1, ad_f^{\kappa_1-1} \tilde{g}_i \rangle \equiv 0, \quad \text{around } x_0, \quad i = 2, \dots, m.$$

For ease of notation we will now omit the tildes above g_i , and thus denote $\tilde{g}_2, \dots, \tilde{g}_m$ again by g_2, \dots, g_m .

Step $k+1$. Assume that the functions $\beta_i^j := \langle d\tilde{h}_j, ad_f^{\kappa_j-1} g_i \rangle$ satisfy

$$(A1) \quad \beta_j^j(x_0) \neq 0, \quad j = 1, \dots, k$$

$$(A2) \quad \beta_i^j \equiv 0, \quad i = j+1, \dots, m, \quad j = 1, \dots, k.$$

We will show that, after applying feedback, (A1) and (A2) also hold for $j = k + 1$. First we note that since $\langle d\tilde{h}_j, G_{\kappa_j-1} \rangle = 0$ (cf. (2.23a)) repeated use of the Leibniz rule yields

$$(2.32) \quad \langle dL_f^\ell \tilde{h}_j, ad_f^{\kappa_j-\ell-1} g_i \rangle = (-1)^\ell \beta_i^j, \quad \ell = 0, 1, \dots, \kappa_j - 1, \quad i, j = 1, \dots, m.$$

By using (A1) and (A2) this implies that we have the following “table” for the expressions $\langle dL_f^{\kappa_1-\kappa_k} \tilde{h}_j, ad_f^{\kappa_k-1} g_i \rangle, j = 1, \dots, k, i = 1, \dots, m$:

$$(2.33) \quad \begin{array}{ccccccc} & & ad_f^{\kappa_k-1} g_1 & ad_f^{\kappa_k-1} g_2 & \dots & ad_f^{\kappa_k-1} g_k & \dots & ad_f^{\kappa_k-1} g_m \\ dL_f^{\kappa_1-\kappa_k} \tilde{h}_1 & & * & & & 0 & & \\ dL_f^{\kappa_2-\kappa_k} \tilde{h}_2 & & & * & & & & \\ \vdots & & ? & & & & & 0 \\ d\tilde{h}_k & & & & & * & & \end{array}$$

where the * elements are all nonzero by (A1). It follows that the map $F = (L_f^{\kappa_1-\kappa_k} \tilde{h}_1, L_f^{\kappa_2-\kappa_k} \tilde{h}_2, \dots, \tilde{h}_k)$ has rank k (the same argument is used in feedback linearization, cf. [HSM], [Is], [NvdS]) and furthermore, since $G_{\kappa_k} + \ker dF = TM$, Lemma 5 implies that G_{κ_k} is projectable by F , while

$$(2.34) \quad \dim F_* G_{\kappa_k} = k.$$

Now consider \tilde{h}_{k+1} . Because of (2.23) it follows that there exists some $i \in \{1, \dots, m\}$ such that

$$(2.35) \quad \langle d\tilde{h}_{k+1}, ad_f^{\kappa_{k+1}-1} g_i \rangle(x_0) \neq 0.$$

We *claim* that we can take $i \in \{k+1, \dots, m\}$ having this property. Indeed, otherwise we would have

$$(2.36) \quad \langle d\tilde{h}_{k+1}, ad_f^{\kappa_{k+1}-1} g_i \rangle(x_0) = 0, \quad i = k+1, \dots, m.$$

Now take any $X \in G_{\kappa_{k+1}} \cap \ker dh$, then X is of the form $X = \sum_{i=1}^n \alpha_i ad_f^{\kappa_{k+1}-1} g_i + Z, Z \in G_{\kappa_{k+1}-1}$, and with the functions α_i satisfying

$$(2.37) \quad \begin{aligned} 0 &= \langle d\tilde{h}_{k+1}, \sum_{i=1}^n \alpha_i ad_f^{\kappa_{k+1}-1} g_i \rangle(x_0) \\ &= \sum_{i=1}^k \alpha_i(x_0) c_i, \quad \text{with } c_i := \langle d\tilde{h}_{k+1}, ad_f^{\kappa_{k+1}-1} g_i \rangle(x_0) \end{aligned}$$

where at least one of the c_i 's is unequal to zero because of (2.35). Now

$$G_{\kappa_{k+1}+1} = S_{\kappa_{k+1}+1} = [f, G_{\kappa_{k+1}} \cap \ker dh] + G_{\kappa_{k+1}}$$

and inductively,

$$(2.38) \quad G_{\kappa_k} \subset ad_f^{\kappa_k-\kappa_{k+1}} (G_{\kappa_{k+1}} \cap \ker dh) + G_{\kappa_k-1}.$$

Therefore any element of G_{κ_k} is of the form

$$(2.39) \quad \sum_{i=1}^m \alpha_i ad_f^{\kappa_k-1} g_i + Z, \quad Z \in G_{\kappa_k-1},$$

with $\alpha_i(x_0), i = 1, \dots, k$, satisfying (2.37). Hence, because of table (2.33) and the nontrivial relation (2.37), the space $(F_* G_{\kappa_k})(F(x_0))$ is at most $(k-1)$ dimensional which is in

contradiction with (2.34). Therefore there *does* exist some $i \in \{k+1, \dots, m\}$ such that (2.35) holds. After reordering, if necessary, g_{k+1}, \dots, g_m we may thus assume that

$$(2.40) \quad \langle d\tilde{h}_{k+1}, ad_f^{\kappa_{k+1}-1} g_{k+1} \rangle(x_0) \neq 0.$$

Now define $\beta_i^{k+1} := \langle d\tilde{h}_{k+1}, ad_f^{\kappa_{k+1}-1} g_i \rangle$ and set $\tilde{g}_i := g_i - (\beta_i^{k+1}/\beta_{k+1}^{k+1})g_{k+1}$, $i = k+2, \dots, m$. Then as in Step 1, cf. (2.30), we obtain

$$(2.41) \quad \langle d\tilde{h}_{k+1}, ad_f^{\kappa_{k+1}-1} \tilde{g}_i \rangle \equiv 0, \quad \text{around } x_0, \quad i = k+2, \dots, m.$$

Omitting again the tildes above \tilde{g}_i we have thus proved that (A1), (A2) also hold for $j = k+1$. Hence by induction we have proved that (A1), (A2) hold for every $k = 1, \dots, m$, for the feedback transformed system (the feedback arising from successively applying Step 1 up to Step m). It immediately follows from expression (2.25) that the decoupling matrix $\tilde{D}(x_0)$ for this feedback transformed system is a lower triangular matrix with nonzero diagonal elements $\beta_1^1(x_0), \dots, \beta_m^m(x_0)$, and thus is nonsingular. Since the rank of the decoupling matrix is invariant under feedback [Is], [NvdS] we have proved the claim. \square

As we have already remarked (see Remark 2 after Theorem 4), Theorem 4 and its proof are closely related to the local feedback linearization problem [JR], [HSM]. However, we would like to stress that from a computational point of view the transformation of Σ into a prime system (P) as given by Theorem 4 may be much simpler than the solution to the local feedback linearization problem. In fact for the latter problem we have to find, in some way or another (see [JR], [HSM]), rectifying (Frobenius) coordinates for the whole sequence of distributions $G_1 \subset G_2 \subset \dots \subset G_n = TM$, on the (possibly high-dimensional) state space manifold M . On the other hand, in order to transform Σ into (P) we basically have to find rectifying (Frobenius) coordinates for the projected distributions $H_1 \subset H_2 \subset \dots \subset H_n = TY$ on the output space manifold Y . In general the dimension of Y is much smaller than that of M , and therefore, *assuming* that the projections H_1, \dots, H_n are easily computed, the latter problem is likely to be simpler. We defer a more elaborate computational implementation of Theorem 4 until after the proof of the next theorem, which deals with the more general problem of local equivalence to input-output prime systems. Recall that given two distributions D_1, D_2 on M we call D_1 involutive modulo D_2 if for any two $X, Y \in D_1$ we have $[X, Y] \in D_1 + D_2$. Furthermore, observe that if the codistributions P_i , $i = 1, 2, \dots$ in (2.4) are constant dimensional then V^* , the largest locally controlled distribution contained in $\ker dh$, exists and is constant dimensional (and is given as $V^* = \ker P^*$).

THEOREM 6. *Consider a nonlinear system Σ with equilibrium x_0 . Σ is locally equivalent to an input-output prime system $(I-O-P)$ with equilibrium $(0, z_0)$, if and only if the following conditions are satisfied in a neighborhood of x_0*

- (i) P_i is constant dimensional $i = 1, 2, \dots, n$;
- (ii) G_i is involutive modulo V^* , and $G_i + V^*$ is constant dimensional; $i = 1, \dots, n-1$;
- (iii) $G_n + V^* = TM$;
- (iv) $G_i = S_i$ modulo V^* , $i = 1, 2, \dots, n$;
- (v) $G_i + \ker dh$ is involutive and of constant dimension, $i = 1, 2, \dots, n-1$.

Proof (only if). Suppose that Σ is locally equivalent to $(I-O-P)$. Clearly, $(I-O-P)$ satisfies conditions (i)–(v) (notice that $V^* = \text{span}\{\partial/\partial z\}$). By (2.2) the definition of S_i is invariant under feedback. From the fact that $(I-O-P)$ satisfies condition (ii), it follows that also the definition of G_i is invariant under feedback modulo V^* (i.e., G_i for the feedback transformed system is equal modulo V^* to G_i for the original system). In

particular, since $V^* \subset \ker dh$, it follows that the definition of $G_i + \ker dh$ is invariant under feedback. Thus conditions (i)–(v) are feedback invariant and we can conclude that they are satisfied by Σ .

(If.) By definition of V^* (see, e.g., [IKGM], [Hi]) there exists locally around x_0 a feedback $u = \alpha(x) + \beta(x)v$, $\det \beta(x) \neq 0$, such that

$$(2.42) \quad [\tilde{f}, V^*] \subset V^*, \quad [\tilde{g}_j, V^*] \subset V^*, \quad j = 1, \dots, m,$$

where $\dot{x} = \tilde{f}(x) + \sum_{j=1}^m \tilde{g}_j(x)v_j$ denotes the feedback transformed system. Thus locally around x_0 we can factor out by the distribution V^* to obtain a manifold M' and a factor system (see [IKGM])

$$(2.43) \quad (\Sigma') \quad \begin{aligned} \dot{x}' &= f'(x') + \sum_{j=1}^m g'_j(x')v_j, & x' \in M', \\ y_j &= h'_j(x'), & j = 1, \dots, m, \end{aligned}$$

i.e., around x_0 we have the projection $\pi : M \rightarrow M'$, with $V^* = \ker \pi_*$. (Since $V^* \subset \ker dh$, $h = (h_1, \dots, h_m)$ can be also factored to a map $h' = (h'_1, \dots, h'_m) : M' \rightarrow Y$ satisfying $h = h' \circ \pi$.)

Define the distributions G'_i and S'_i for the factor system Σ' . It is readily checked that G'_i and S'_i satisfy conditions (i)–(iv) of Theorem 4 for the factor system Σ' and around $x'_0 = \pi(x_0)$. Indeed, observe again that under conditions (ii) and (iv) of Theorem 6 the distributions G_i and S_i are feedback invariant modulo V^* . Then it immediately follows that G'_i and S'_i satisfy conditions (i)–(iii) of Theorem 4 applied to Σ' . Finally, since $V^* \subset \ker dh$ it follows that G_i are also feedback invariant modulo $\ker dh$. Thus $\pi_*(G_i + \ker dh) = G'_i + \ker dh'$, and it follows from Lemma 2 (applied to the involutive and constant-dimensional distribution $G_i + \ker dh$ and the mapping $\pi : M \rightarrow M'$) that $G'_i + \ker dh'$ is involutive and constant dimensional.

Hence by Theorem 4, Σ' is locally equivalent to a prime system (P) of the form (1.3a) (with $x'_0 = 0$, and $\mu_i, i = 1, \dots, m$, the controllability indices of Σ'). Since the remaining dynamics of Σ are of the general form (1.3b) we conclude that Σ is locally equivalent to $(I - O - P)$ with equilibrium $x_0 = (0, z_0)$. \square

Remark 1. Note that the indices μ_1, \dots, μ_m are *intrinsically* defined. Indeed if Σ satisfies the conditions of Theorem 6 then μ_1, \dots, μ_m are the (intrinsically defined) controllability indices of the factor system Σ' , living on M/V^* . In particular it follows that an input-output prime system $(I - O - P)$ cannot be equivalent to an input-output prime system with different indices μ_1, \dots, μ_m .

Remark 2. If Σ satisfies the conditions of Theorem 6 on a neighborhood of a point \bar{x} which is *not* an equilibrium, then Σ will be locally equivalent to an input-output prime system (1.4) with the addition of a constant drift term $f(\bar{x})$. Furthermore, if $f(\bar{x}) \in G(\bar{x})$ then this drift term can be removed by additional feedback. Similarly, if $h(\bar{x}) \neq 0$, then we have to add to the output equation of (1.4a) the constant term $h(\bar{x})$. Of course, this remark already applies to Theorem 4.

Remark 3. It follows from the proof of Theorem 6 that h_*G_i is a well-defined distribution on a neighborhood of $y_0 = h(x_0)$ (i.e., G_i is projectable by h on some neighborhood V_{x_0}), $i = 1, \dots, m$. In fact $h_*G_i = h'_*G'_i$ (with $'$ denoting the factor system Σ'), and the projectability of G'_i by h' to an involutive constant-dimensional distribution on a neighborhood of y_0 follows by an application of Lemma 5 to G'_i and h' . Note, however, that

Lemma 5 as it stands *cannot* be directly applied to G_i and h (satisfying condition (v)), since we do not require G_i to be involutive and constant dimensional (but only modulo V^*).

Note that Theorem 6 generalizes the well-known fact that a nonlinear system Σ whose decoupling matrix $D(x)$ (cf. (2.6)) has rank m around x_0 can be transformed by local state space and feedback transformations into (1.3), see, e.g., [IKGM]. Hence Theorem 6 can also be interpreted as giving the necessary and sufficient conditions for finding a local output transformation $\tilde{y} = \psi(y) = (\psi_1(y), \dots, \psi_m(y))$ such that the decoupling matrix $\tilde{D}(x)$ for the transformed output functions $\tilde{h}_1 = \psi_1 \circ h, \dots, \tilde{h}_m = \psi_m \circ h$ has rank m around x_0 .

Example. Consider the following system on $M = \mathbb{R}^3, Y = \mathbb{R}^2$:

$$(2.44) \quad \begin{aligned} \dot{x}_1 &= u_2, & y_1 &= x_1 \\ \dot{x}_2 &= x_3, & y_2 &= x_2 + \frac{1}{3}x_1^3 \\ \dot{x}_3 &= u_1. \end{aligned}$$

The relative degrees are both 1, while the decoupling matrix $D(x)$ equals

$$D(x) = \begin{pmatrix} 0 & 1 \\ 0 & x_1^2 \end{pmatrix}$$

and thus is singular, implying that the system is *not* input-output decouplable by static state feedback. However it is readily seen that the system satisfies the conditions of Theorem 6 and even of Theorem 4, and in fact we only need the output transformation

$$\psi_1(y) = y_2 - \frac{1}{3}y_1^3, \quad \psi_2(y) = y_1$$

to bring the system into prime form (1.2), with $\kappa_1 = 1, \kappa_2 = 2$ (being the relative degrees of the *transformed* system)! Now suppose we want to asymptotically track a desired smooth trajectory $y^d(t) = (y_1^d(t), y_2^d(t)), t \geq 0$ for (2.44). Using the above output transformation, such a trajectory is transformed into the new coordinates as $\tilde{y}^d(t) = (y_2^d(t) - \frac{1}{3}(y_1^d(t))^3, y_1^d(t)), t \geq 0$, and since (2.44) has been transformed into a prime system the tracking problem is simply solved by a control strategy which is *linear* in the transformed coordinates, namely,

$$\begin{aligned} u_1 &= -K_{11}(x_2 - (y_2^d(t) - \frac{1}{3}(y_1^d(t))^3) \\ &\quad - K_{12}(x_3 - (\dot{y}_2^d(t) - (y_1^d(t))^2 \dot{y}_1^d(t))) \\ &\quad + \ddot{y}_2^d(t) - 2y_1^d(t)(\dot{y}_1^d(t))^2 - (y_1^d(t))^2 \ddot{y}_1^d(t) \\ u_2 &= -K_2(x_1 - y_1^d(t)) + \dot{y}_1^d(t) \end{aligned}$$

where $K_2 < 0$, and K_{11}, K_{12} are designed in such a way that the polynomial $s^2 + K_{12}s + K_{11}$ is Hurwitz.

Notice, furthermore, that the conditions of Theorem 6 imply (see, e.g., [NvdS], [Is]) that Σ is input-output decouplable by *dynamic* state feedback. (In the foregoing example, system (2.44) can be dynamically input-output decoupled for the original output functions by pre-integrating the input u_2 one time). Regarded from this viewpoint, Theorem 6 avoids the addition of extra pre-integrators to the system by allowing instead for output transformations.

The proofs of Theorems 4 and 6 immediately yield the following algorithm to transform Σ into a prime or input-output prime system.

ALGORITHM 7. Consider a nonlinear system Σ with equilibrium x_0 , and satisfying, in a neighborhood V_{x_0} of x_0 , conditions (i)–(v) of Theorem 6. Then Σ can be transformed into $(I - O - P)$ in the following way.

(a) Compute the distributions $H_i := h_* G_i$ on $W_{y_0} = h(V_{x_0})$, $i = 1, 2, \dots, n-1$. (By Remark 3 above, H_i are all well defined, involutive, and constant-dimensional distributions.)

(b) Construct rectifying (Frobenius) coordinates ψ_1, \dots, ψ_m (cf. (2.18), (2.19)) defined on a possibly smaller neighborhood of y_0 , for the whole sequence $H_1 \subset H_2 \subset \dots \subset H_{n-1}$. This defines the output space transformation ψ of Definition 1.

(c) Consider the output functions $\tilde{h}_i := \psi_i \circ h$, $i = 1, \dots, m$, for Σ . Compute the relative degrees μ_1, \dots, μ_m for these output functions and the decoupling matrix $\tilde{D}(x) = (L_{g_j} L_f^{\mu_i - 1} \tilde{h}_i(x))_{i,j=1,\dots,m}$. Necessarily $\mu_i < \infty$, $i = 1, \dots, m$, and $\text{rank } \tilde{D}(x) = m$ around x_0 . Define the functions

$$(2.45) \quad x_{ij} := L_f^{(j-1)} \tilde{h}_i, \quad j = 1, \dots, \mu_i, \quad i = 1, \dots, m.$$

Necessarily these functions are independent around x_0 , while

$$(2.46) \quad V^* = \ker \text{span}\{dx_{ij}, j = 1, \dots, \mu_i, \quad i = 1, \dots, m\}.$$

Choose complementary coordinates $z = (z_1, \dots, z_n)$ around x_0 ($n' := n - (\mu_1 + \dots + \mu_m)$). This defines the state space transformation φ of Definition 1.

(d) Compute the regular feedback $u = \alpha(x) + \beta(x)v$ around x_0 as

$$(2.47) \quad \alpha(x) = -\tilde{D}^{-1}(x) \begin{pmatrix} L_f^{\mu_1} \tilde{h}_1(x) \\ \vdots \\ L_f^{\mu_m} \tilde{h}_m(x) \end{pmatrix}, \quad \beta(x) = \tilde{D}^{-1}(x).$$

This defines the feedback transformation required in Definition 1.

Remark 1. Note that $\mu_1 + \dots + \mu_m = n$ if and only if $V^* = 0$, in which case Σ is locally equivalent to a prime system.

Remark 2. At some occasions it may be more efficient *not* to check conditions (i)–(v) of Theorem 6 in order to see if Σ is locally equivalent to $(I - O - P)$, but instead to apply *directly* Algorithm 7. If the Algorithm breaks down (e.g., if some distributions H_i are not well defined or not involutive, or if $\tilde{D}(x)$ does not have full rank) then Σ is not locally equivalent to $(I - O - P)$ (while Σ is locally equivalent to $I - O - P$ if Algorithm 7 *does* work).

Example. As an illustration of the above remark we apply Algorithm 7 to the example

$$\begin{aligned} \dot{x}_1 &= u_1, & y_1 &= x_5 + x_5 x_3 - \frac{1}{2} x_5 x_2^2 \\ \dot{x}_2 &= u_2, & y_2 &= x_3 - \frac{1}{2} x_2^2 \\ \dot{x}_3 &= x_1 + x_2 u_2 \\ \dot{x}_4 &= x_2 + x_1 u_1 \\ \dot{x}_5 &= x_4 - \frac{1}{2} x_1^2. \end{aligned}$$

Following the proof of Lemma 5 we first express the system in local coordinates

$$\xi_1 = x_1, \quad \xi_2 = x_2, \quad \xi_3 = x_3 - \frac{1}{2} x_2^2, \quad \xi_4 = x_4, \quad \xi_5 = x_5(1 + x_3 - \frac{1}{2} x_2^2)$$

as

$$\begin{aligned}\dot{\xi}_1 &= u_1, & y_1 &= \xi_5 \\ \dot{\xi}_2 &= u_2, & y_2 &= \xi_3 \\ \dot{\xi}_3 &= \xi_1 \\ \dot{\xi}_4 &= \xi_2 + \xi_1 u_1 \\ \dot{\xi}_5 &= \frac{\xi_5}{1 + \xi_3}(1 + \xi_1) + (\xi_4 - \frac{1}{2}\xi_1^2)(1 + \xi_3).\end{aligned}$$

(a) Simple computations give

$$\begin{aligned}G_1 &= \text{span} \left\{ \frac{\partial}{\partial \xi_1}, +\xi_1 \frac{\partial}{\partial \xi_4}, \frac{\partial}{\partial \xi_2} \right\} \\ G_2 &= \text{span} \left\{ \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_4}, \frac{\partial}{\partial \xi_3} + \frac{\xi_5}{1 + \xi_3} \frac{\partial}{\partial \xi_5} \right\} \\ G_3 &= TM \\ H_1 &= 0 \\ H_2 &= \text{span} \left\{ \frac{y_1}{1 + y_2} \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right\} \\ H_3 &= TY.\end{aligned}$$

According to (2.15) the indices are $\kappa_2 = 2, \kappa_1 = 3$.

(b) From (2.18) the rectifying Frobenius coordinates are

$$\psi_1(y) = \frac{y_1}{1 + y_2} \quad \psi_2(y) = y_2.$$

(c) The transformed output functions \tilde{h}_1, \tilde{h}_2 are

$$\tilde{h}_1 = \frac{x_5(1 + x_3 - \frac{1}{2}x_2^2)}{1 + x_3 - \frac{1}{2}x_2^2} = x_5 \quad \tilde{h}_2 = x_3 - \frac{1}{2}x_2^2.$$

The relative degrees are $\mu_1 = 3, \mu_2 = 2$, while the decoupling matrix is $\tilde{D} = I_{2 \times 2}$. Hence the functions

$$\begin{aligned}z_1 &= \tilde{h}_1 = x_5 \\ z_2 &= L_f \tilde{h}_1 = x_4 - \frac{1}{2}x_1^2 \\ z_3 &= L_f^2 \tilde{h}_1 = x_2 \\ z_4 &= \tilde{h}_2 = x_3 - \frac{1}{2}x_2^2 \\ z_5 &= L_f \tilde{h}_2 = x_1\end{aligned}$$

give the state space transformation φ of Definition 1.

(d) The regular feedback of definition 1 is

$$\alpha(x) = 0 \quad \beta(x) = I.$$

In fact, in z -coordinates we have

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = u_2 \quad \dot{z}_4 = z_5 \quad \dot{z}_5 = u_1,$$

which is a linear prime system.

The extension of Theorem 6 to local equivalence into input-output prime systems of special form (1.3) reads as follows.

PROPOSITION 8. *Consider a nonlinear system Σ with equilibrium x_0 . Σ is locally equivalent to an input-output prime system of special form $(I - O - P - S)$, if and only if, on a neighborhood of x_0 , conditions (i)–(v) of Theorem 6 are satisfied and, additionally:*

(vi) $V^* \cap S^* = 0$;

(vii) S^* is involutive and constant dimensional.

Proof (only if). Suppose Σ is locally equivalent to $(I - O - P - S)$. From the (only if) part of Theorem 6 it follows that Σ satisfies conditions (i)–(v). Clearly, $(I - O - P - S)$ satisfies conditions (vi) and (vii). Furthermore the definition of S^* is feedback invariant, and thus also Σ satisfies conditions (vi) and (vii).

(If.) By (iii), (iv), and (vi) we have $V^* \oplus S^* = TM$. By Theorem 6, Σ is locally equivalent to $(I - O - P)$, i.e., (1.3). Here z are coordinate functions which are arbitrary except for the fact that they have to be complementary to the coordinate functions $x = (x_{11}, \dots, x_{m\mu_m})$; see (2.45). In the present case, however, since $V^* \oplus S^* = TM$ and V^* and S^* are involutive and constant dimensional, we can choose z such that $\text{span } dz = \text{ann } S^*$. Since, by construction, $\text{span } dx = \text{ann } V^*$, cf. (2.46), we thus have

$$(2.48) \quad V^* = \text{span} \left\{ \frac{\partial}{\partial z} \right\}, \quad S^* = \text{span} \left\{ \frac{\partial}{\partial x} \right\}.$$

Then, first of all, since $G = S_1 \subset S^* = \text{span} \{ \partial/\partial x \}$, we have in (1.3)

$$(2.49) \quad b_j(z, x) = 0, \quad j = 1, \dots, m.$$

Second, by definition of S^* , $[f, S^* \cap \ker dh] \subset S^*$, cf. (2.3), and thus, since f is of the form $f = * \partial/\partial x + a(z, x) \partial/\partial z$ and $S^* = \text{span} \{ \partial/\partial x \}$

$$(2.50) \quad \left[* \frac{\partial}{\partial x} + a(z, x) \frac{\partial}{\partial z}, \frac{\partial}{\partial x_{ij}} \right] \subset \text{span} \left\{ \frac{\partial}{\partial x} \right\}, \quad j = 2, \dots, \mu_i, \quad i = 1, \dots, m.$$

(Note that $\ker dh$ is everything minus $\text{span} \{ (\partial/\partial x_{i1}), i = 1, \dots, m \}$.) This implies that $a(z, x)$ in (1.3) may only depend on z and $x_{i1} = y_i, i = 1, \dots, m$, and thus (1.4) results. \square

Remark 1. If conditions (ii) and (iv) in Proposition 8 are replaced by the stronger conditions

(ii)' G_i is involutive and of constant dimension, $i = 1, 2, \dots, n - 1$,

(iv)' $G_i = S_i, i = 1, \dots, n$,

then, following [MBE], $a(z, y)$ in (1.4b) will only depend on those (new) output components y_i with i such that $\mu_i = \max \{ \mu_1, \dots, \mu_m \}$.

Remark 2. Necessary and sufficient conditions for transforming into (1.4) (without change of output space coordinates) a nonlinear system Σ having invertible decoupling matrix have been identified in [BI]: see also [MBE]. Similar conditions were derived, in a different context, in [vdS]. Notice that in the *linear* case condition (vii) is automatically satisfied. This explains that for a linear system we can always write (even if condition (vi) is not satisfied) the V^* dynamics as being only driven by y , as follows from the Morse canonical form [Mo].

EXAMPLE. Consider the single input system

$$\dot{x}_1 = x_4^3, \quad y = x_2$$

$$\begin{aligned}\dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u.\end{aligned}$$

Easy computations give:

$$\begin{aligned}V^* &= \text{span} \left\{ \frac{\partial}{\partial x_1} \right\}, \quad \ker dh = \text{span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\}, \quad S^* = \mathbb{R}^4, \\ S^* \cap V^* &= \text{span} \left\{ \frac{\partial}{\partial x_1} \right\}, \\ G_1 &= \text{span} \left\{ \frac{\partial}{\partial x_4} \right\}, \quad G_2 = \text{span} \left\{ \frac{\partial}{\partial x_4}, 3x_4^2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right\}, \\ G_3 &= \text{span} \left\{ \frac{\partial}{\partial x_4}, 3x_4^2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} \right\}.\end{aligned}$$

Theorem 4 does not apply since G_2 is not involutive. Theorem 6 applies, while Proposition 8 does not apply since $V^* \cap S^* \neq 0$.

Now let us proceed to a *global* version of the equivalence results we have obtained so far. Instead of requiring that $\text{rank } dh(x)$ and $\dim G(x)$ are equal to m in a neighborhood of x_0 , we will now have to require this on the whole M . Then by the Rank Theorem (e.g., [Sp]), $h(M)$ is an open part of Y , and without loss of generality, we may restrict to this part of Y and assume that $h : M \rightarrow Y$ is surjective. The global version of Definition 1 reads now as follows.

DEFINITION 9. Consider two systems Σ_1, Σ_2 defined on $(M_1, Y_1), (M_2, Y_2)$ with equilibrium points $x_{01} \in M_1, x_{02} \in M_2$, respectively. Σ_1 is globally equivalent to Σ_2 if there exist:

- (i) A diffeomorphism $\varphi : M_1 \rightarrow M_2$, satisfying $\varphi(x_{01}) = x_{02}$;
- (ii) a nonsingular feedback $u = \alpha(x) + \beta(x)v$ on M_1 with $\alpha(x_{01}) = 0$ and $\det \beta(x) \neq 0$;
- (iii) a diffeomorphism $\psi : Y_1 \rightarrow Y_2$ with $\psi(h_1(x_{01})) = h_2(x_{02})$ such that the resulting transformation of Σ_1 equals Σ_2 .

Since Theorem 6 generalizes Theorem 4 we will only give the global version of Theorem 6, and state as a corollary the global version of Proposition 8.

THEOREM 10. Consider a nonlinear system Σ on (M, Y) with equilibrium x_0 , and assume that $h : M \rightarrow Y$ is a surjective submersion and that $\dim G(x) = m$, for all $x \in M$. Suppose that conditions (i)–(v) of Theorem 6 are satisfied on the whole M , and that

- (A) There exist globally defined independent functions ψ_1, \dots, ψ_m on Y which are rectifying coordinates for H_1, \dots, H_m , i.e., (2.19a) and (2.19b) are satisfied for every $x \in M$ (the local existence of ψ_1, \dots, ψ_m is already insured by conditions (i)–(v));

then by Algorithm 7(c), (d) V^* is globally given by (2.46) and the feedback (2.47) is globally defined. Furthermore, there exists a surjective submersion $\pi : M \rightarrow M'$ with $\ker \pi_* = V^*$, while the factor system Σ' , cf. (2.43), is globally defined on M' .

Assume additionally that

- (B) The vectorfields f' and $g'_j, j = 1, \dots, m$, on M' , cf. (2.43), are complete;

then M' equals $\mathbb{R}^\mu, \mu = \sum_{i=1}^m \mu_i$, and thus Σ is globally equivalent to an input-output system $(I - O - P)$ with equilibrium $(0, z_0)$. Conversely, if Σ is globally equivalent to

($I - O - P$) then conditions (i)–(v) of Theorem 6 are satisfied on the whole M , and conditions (A) and (B) hold.

Remark. Since the feedback (α, β) depends on the choice of ψ_1, \dots, ψ_m , also condition (B), i.e., the completeness of the modified vectorfields $f', g'_j, j = 1, \dots, m$, may depend on the choice of ψ_1, \dots, ψ_m . This is already illustrated by the following very simple example: Consider the system $\dot{x} = u, y = e^{-x}$ on $M = \mathbb{R}$ and $Y = (0, \infty)$. If we take $\psi(y) = \ln y$ as a global coordinate on Y (which trivially is rectifying, since $H_1 = TY$) then $\tilde{h}(x) = \psi \circ h(x) = -x$, and $g' = \tilde{g} = -\partial/\partial x$ is complete, implying that the system is globally equivalent to the prime system $\dot{x} = u, y = x$. However if we would take the global rectifying coordinate $\psi(y) = y$, then $g' = e^x \partial/\partial x$ is not complete, and indeed, since $\text{id} : (0, \infty) \rightarrow \mathbb{R}$ is not a diffeomorphism onto \mathbb{R} , the system is not globally transformed into a prime system.

Proof. Suppose conditions (i)–(v) are satisfied on M , as well as condition (A). Apply Algorithm 7 using the global rectifying coordinates ψ_1, \dots, ψ_m on Y . Since V^* is constant dimensional on the whole M it follows by a slight adaptation of [HK, Thm. 3.9], see also [IKGM], that V^* can be globally factored out, i.e., there exists a surjective submersion $\pi : M \rightarrow M'$ with $\ker \pi_* = V^*$, and the feedback transformed dynamics $\tilde{f}, \tilde{g}_j, j = 1, \dots, m$ (with (α, β) defined by (2.47)) project to dynamics $f', g'_j, j = 1, \dots, m$ on M' (note that, in contrast to [HK, Thm. 3.9], we do not require Σ to be accessible; however, condition (iii) of Theorem 6 insures that Σ is “accessible modulo V^* ”). Now assume that condition (B) is satisfied. By the local equivalence of Σ with (1.3) it follows that the vectorfields $g'_j, \text{ad}_f g'_j, \dots, \text{ad}_f^{\mu_j-1} g'_j, j = 1, \dots, m$, are commuting and complete vectorfields on M' (see [Re], [DBE]). It follows that $M' = \mathbb{R}^{\mu-k} \times S^k$ for some $k \geq 0$. However, since the functions $L_f^j \tilde{h}_i, j = 0, 1, \dots, \mu_i - 1, i = 1, \dots, m$, are global coordinate functions on M' necessarily $k = 0$ (since S^k is compact). Since $y_j = x_{j1}, j = 1, \dots, m$, we also have $Y \simeq \mathbb{R}^m$. It follows [Re], [DBE] that Σ' is globally equivalent to a linear system, and thus that Σ is globally equivalent to $I - O - P$. Conversely, if Σ is globally equivalent to ($I - O - P$) then by the (only if) part of Theorem 6 conditions (i)–(v) are satisfied everywhere. Furthermore, clearly ($I - O - P$) satisfies Conditions A and B. \square

COROLLARY 11. Σ is globally equivalent to ($I - O - P - S$) if and only if, in addition to conditions (i)–(v) and conditions A and B of Theorem 10, conditions (vi), (vii) of Proposition 8 are satisfied on the whole M .

Remark. Analogous reasoning on the global equivalence modulo V^* to a linear system was used in [MRS]. Similar conditions for the global equivalence of a nonlinear system with invertible decoupling matrix into (1.3) or (1.4) were derived in [BI].

3. Conclusions and final remarks. Necessary and sufficient geometric conditions have been given for transforming nonlinear systems into (input-output) prime form (of special form), locally as well as globally. The main novelty (e.g., as compared to normal forms for input-output decouplable systems) is that we allow for output transformations. Actually, as made explicit in Algorithm 7 (see also the example following it), the output transformation is the crucial step in the whole transformation procedure and involves the simultaneous integration of a nested sequence of distributions on the output space manifold (similar to the integration of distributions on the state space manifold as in the feedback linearization problem). The results obtained are applicable to control problems where output transformations are naturally allowed, such as output tracking, output regulation, (almost) disturbance decoupling [I], [NvdS], [MRS] and the servomechanism problem. The results enable us to treat the class of nonlinear systems equivalent to input-output prime form very much like the well-studied class of input-output decouplable systems. Finally, as we

have remarked, the use of output transformations may be an alternative to the use of extra pre-integrators for *dynamic* input-output decoupling. This raises the problem of how output transformations may be used to *minimize* the amount of pre-integrators for dynamic input-output decoupling.

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